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Local Smoothness of Functions and Baskakov–Durrmeyer Operators

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In this paper, we consider the local approximation by Baskakov–Durrmeyer operators. The continuous functions of local Lipschitz-α $(0<\alpha<1)$ on any subset of $[0, \infty)$ are characterized by the local rate of convergence of Baskakov–Durrmeyer operators. The main difference between these operators and their classical and Kantorovich-variants respectively is that they have commutativity, which is crucial for our purpose. © 1997 Academic Press

I. INTRODUCTION AND MAIN RESULTS

The Baskakov–Durrmeyer operators on $[0, \infty)$ are defined as

$$V_n(f, x)=(n-1)\sum_{k=0}^{\infty} v_{n,k}(x)\int_0^{\infty} v_{n,k}(t)\, f(t)\, dt, \qquad x\in[0, \infty), \quad (1.1)$$

where f is a function for which the right side of (1.1) makes sense and

$$v_{n,k}(x)=\binom{n+k-1}{k} x^k(1+x)^{-n-k}, \qquad 1<n\in N.$$

These operators are very interesting approximation processes and have many nice properties such as commutativity. Their approximation rates are closely related to the smooth property of the function they approximate [3, 4, 9].

In 1991, Zhou [9] used the Baskakov–Durrmeyer operators to characterize the lip α functions on $[0, \infty)$ with $0<\alpha<1$. In fact, it was proved in [9] that for $f\in C[0, \infty)\cap L_\infty[0, \infty)$, $0<\alpha<1$,

$$|V_n(f, x)-f(x)|\leqslant M_f\left(\frac{x(1+x)}{n}+\frac{1}{n^2}\right)^{\alpha/2}, \qquad 1<n\in N, \quad x\in[0, \infty), \qquad (1.2)$$

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with a constant M_f independent of n and x, if and only if $\omega_1(f, t) = 0(t^a)$, where

$$\omega_1(f, t) = \sup_{0 < h \leqslant t} \|\Delta_h f(x)\|_\infty,$$

$$\Delta_h f(x) = f(x+h) - f(x), \qquad \text{if} \quad x, x+h \in [0, \infty); \tag{1.3}$$

$$\Delta_h f(x) = 0, \qquad \text{otherwise.}$$

Such an equivalence was first given by Berens and Lorentz [1] for Bernstein operators in 1972. Some more characterizations of second orders and higher orders of global smoothness of functions along this line can be found in [6, 8] and [10, 11] respectively.

All of the above mentioned results concern the global smoothness of functions and the global approximating property. The purpose of this paper is to give an equivalence between local smoothness of functions and local convergence of Baskakov–Durrmeyer operators as follows.

THEOREM. *Let* $f \in C[0, \infty) \cap L_\infty[0, \infty)$, $V_n(f, x)$ *be given by* (1.1), $0 < \alpha < 1$, *and* E *be any subset of* $[0, \infty)$. *Then we have*

$$|f(x) - f(y)| \leqslant M_f |x - y|^a, \qquad x \in [0, \infty), \quad y \in E, \tag{1.4}$$

if and only if

$$|V_n(f, x) - f(x)| \leqslant M'_f \left(\left(\frac{x(1+x)}{n} + \frac{1}{n^2} \right)^{\alpha/2} + (d(x, E))^a \right),$$

$$x \in [0, \infty), \qquad 3 < n \in N, \tag{1.5}$$

where M_f and M'_f are constants depending only on α and f, $d(x, E)$ is the distance between x and E defined as

$$d(x, E) = \inf_{y \in E} \{|x - y|\}. \tag{1.6}$$

As a consequence, we obtain the early equivalence (1.2).

We say that a continuous function f is locally Lip $\alpha (0 < \alpha \leqslant 1)$ on E if it satisfies the condition (1.4). In particular, when $E = \{x_0\}$, another interesting corollary of our theorem can be stated as follows:

COROLLARY. *Let* $x_0 \in [0, \infty)$, $0 < \alpha < 1$, $f \in C[0, \infty)$. *Then we have*

$$|f(x) - f(x_0)| \leqslant M_f |x - x_0|^a, \qquad x \in [0, \infty), \tag{1.7}$$

if and only if

$$|V_n(f, x) - f(x)| \leqslant M'_{f, x_0}(n^{-\beta} + |x - x_0|^\alpha), \qquad x \in [0, \infty), \tag{1.8}$$

where

$$\beta = \begin{cases} \dfrac{\alpha}{2}, & \text{if} \quad x_0 > 0, \\ \alpha, & \text{if} \quad x_0 = 0. \end{cases}$$

Thus we give another view of the fact that the Bernstein-type positive linear operators have better approximation property at the end points. This case also has many connections to the research of singular detection [7] and to regularity of solutions to PDES [2] and wavelets [5].

In the following sections we shall prove our main result. First, we need some lemma, which presents the Bernstein-type inequalities.

2. LOCAL BERNSTEIN-TYPE INEQUALITIES

To prove the theorem, we need some preliminary results. By simple computations we have the moments of the Baskakov–Durrmeyer operators.

LEMMA 2.1. *Let $V_n(f, x) = V_n(f(t), x)$ be given as (1.1). Then we have for $x \in [0, \infty)$,*

$$V_n(1, x) = 1;$$

$$V_n(t, x) = \frac{nx}{n-2} + \frac{1}{n-2};$$

$$V_n(t^2, x) = \frac{2}{(n-2)(n-3)} + \frac{4nx}{(n-2)(n-3)} + \frac{n(n+1)x^2}{(n-2)(n-3)}.$$

Hence,

$$V_n((t-x)^2, x) \leqslant M\left(\frac{x(1+x)}{n} + \frac{1}{n^2}\right) \tag{2.1}$$

and

$$|V_n(\varphi^2(t), x) - \varphi^2(x)| \leqslant M\left(\frac{x(1+x)}{n} + \frac{1}{n^2}\right), \tag{2.2}$$

where M is independent of n, $n \in N$, and x, $\varphi^2(t) = t(1+t)$.

In the proof of the inverse part we need some Bernstein–Markov-type inequalities as follows.

LEMMA 2.2. *Let $f \in C[0, \infty)$. Then we have*

$$V_n'(f, x) = (n-1) \sum_{k=0}^{\infty} \left(\int_0^{\infty} f(t)\, v_{n,k}(t)\, dt \right) \frac{k-nx}{x(1+x)}\, v_{n,k}(x) \tag{2.3}$$

$$= n(n-1) \sum_{k=0}^{\infty} v_{n+1,k}(x) \int_0^{\infty} (v_{n,k+1}(t) - v_{n,k}(t))\, f(t)\, dt. \tag{2.4}$$

We denote $v_{n,k}(x) = 0$ for $k < 0$. Then we have

$$v_{n,k}'(x) = \frac{k-nx}{x(1+x)}\, v_{n,k}(x).$$

Note that $v_{n,k}'(x) = n(v_{n+1,k-1}(x) - v_{n+1,k}(x))$. The proof of Lemma 2.2 is quite easy and we omit it here.

With these preparations, we can present our local Bernstein-type inequalities as follows.

LEMMA 2.3. *Let $0 < \alpha \leqslant 1$, $E \subset [0, \infty)$. Suppose that $f \in C[0, \infty)$ satisfies*

$$|f(t)| \leqslant \left(\frac{t(t+1)}{n} \right)^{\alpha/2} + n^{-\alpha} + (d(t, E))^{\alpha}, \qquad t \in [0, \infty). \tag{2.5}$$

Then we have

$$|V_n'(f, x)| \leqslant M' \sqrt{\frac{n}{x(1+x)}} \left\{ \left(\frac{x(1+x)}{n} \right)^{\alpha/2} + n^{-\alpha} + (d(x, E))^{\alpha} \right\}, \qquad x \in [0, \infty), \tag{2.6}$$

where M' is independent of f, n, and x.

Proof. We note that for $a, b \geqslant 0$, $0 < \beta \leqslant 1$,

$$(a+b)^{\beta} \leqslant a^{\beta} + b^{\beta}, \tag{2.7}$$

and for $t, x \in [0, \infty)$,

$$d(t, E) \leqslant d(x, E) + |t - x|, \tag{2.8}$$

$$t(1+t) = x(1+x) + 2x(t-x) + (t-x)^2 + (t-x). \tag{2.9}$$

Using the above notes we have from the condition (2.5)

$$|V_n'(f,x)| = \left| \sum_{k=0}^{\infty} \frac{k-nx}{x(1+x)} v_{n,k}(x)(n-1) \int_0^{\infty} f(t)\, v_{n,k}(t)\, dt \right|$$

$$\leqslant \sum_{k=0}^{\infty} \frac{k-nx}{x(1+x)} v_{n,k}(x)(n-1) \int_0^{\infty} v_{n,k}(t) \left\{ \left(\frac{x(1+x)}{n} \right)^{\alpha/2} + n^{-\alpha} \right.$$

$$\left. + (d(x,E))^{\alpha} + 2\,|t-x|^{\alpha} + \left(\frac{|t-x|}{n} \right)^{\alpha/2} + \left(\frac{2x\,|t-x|}{n} \right)^{\alpha/2} \right\} dt$$

$$\leqslant \left\{ \left(\frac{x(1+x)}{n} \right)^{\alpha/2} + n^{-\alpha} + (d(x,E))^{\alpha} \right\}$$

$$\times \frac{1}{x(1+x)} \left(\sum_{k=0}^{\infty} (k-nx)^2\, v_{n,k}(x) \right)^{1/2}$$

$$+ \frac{2}{x(1+x)} \sum_{k=0}^{\infty} |k-nx|\, v_{n,k}(x)(n-1) \left\{ \int_0^{\infty} v_{n,k}(t)(t-x)^2\, dt \right\}^{\alpha/2}$$

$$\times \left\{ \int_0^{\infty} v_{n,k}(t)\, dt \right\}^{1-(\alpha/2)} + \frac{1}{2x(1+x)}\, n^{-\alpha} \left\{ \sum_{k=0}^{\infty} (k-nx)^2\, v_{n,k}(x) \right\}^{1/2}$$

$$+ \frac{1}{2x(1+x)} \sum_{k=0}^{\infty} |k-nx|\, v_{n,k}(x)(n-1)$$

$$\times \left\{ \int_0^{\infty} v_{n,k}(t)(t-x)^2\, dt \right\}^{\alpha/2}$$

$$\times \left\{ \int_0^{\infty} v_{n,k}(t)\, dt \right\}^{1-(\alpha/2)} + \left(\frac{2x}{n} \right)^{\alpha/2} \sum_{k=0}^{\infty} \frac{|k-nx|}{x(1+x)} v_{n,k}(x)(n-1)$$

$$\times \left\{ \int_0^{\infty} v_{n,k}(t)(t-x)^2\, dt \right\}^{\alpha/4} \left\{ \int_0^{\infty} v_{n,k}(t)\, dt \right\}^{1-(\alpha/4)}$$

$$\leqslant \left\{ \left(\frac{x(1+x)}{n} \right)^{\alpha/2} + n^{-\alpha} + (d(x,E))^{\alpha} \right\} \sqrt{\frac{n}{x(1+x)}} + \frac{5}{2}\, \frac{1}{x(1+x)}$$

$$\times \sum_{k=0}^{\infty} |k-nx|\, v_{n,k}(t) \left\{ (n-1) \int_0^{\infty} v_{n,k}(t)(t-x)^2\, dt \right\}^{\alpha/2}$$

$$+ \frac{1}{2x(1+x)}\, n^{-\alpha} \sqrt{nx(1+x)} + \left(\frac{2x}{n} \right)^{\alpha/2}$$

$$\times \sum_{k=0}^{\infty} \frac{|k-nx|}{x(1+x)} v_{n,k}(x) \left\{ (n-1) \int_0^{\infty} v_{n,k}(t)(t-x)^2\, dt \right\}^{\alpha/4}$$

$$
\begin{aligned}
&\leqslant \left\{ \left(\frac{x(1+x)}{n} \right)^{\alpha/2} + \frac{3}{2} n^{-\alpha} + (d(x, E))^{\alpha} \right\} \sqrt{\frac{n}{x(1+x)}} \\
&\quad + \frac{5}{2} \frac{1}{x(1+x)} \left\{ \sum_{k=0}^{\infty} |k-nx|^{1/(1-\alpha/2)} v_{n,k}(x) \right\}^{1-\alpha/2} \\
&\quad \times \left\{ \sum_{k=0}^{\infty} v_{n,k}(x)(n-1) \int_0^{\infty} v_{n,k}(t)(t-x)^2 \, dt \right\}^{\alpha/2} \\
&\quad + \frac{(2x/n)^{\alpha/2}}{x(1+x)} \left\{ \sum_{k=0}^{\infty} |k-nx|^{1-\alpha/4} v_{n,k}(x) \right\}^{1-\alpha/4} \\
&\quad \times \left\{ \sum_{k=0}^{\infty} v_{n,k}(x)(n-1) \int_0^{\infty} v_{n,k}(t)(t-x)^2 \, dt \right\}^{\alpha/4} \\
&\leqslant \left\{ \left(\frac{x(1+x)}{n} \right)^{\alpha/2} + \frac{3}{2} n^{-\alpha} + (d(x, E))^{\alpha} \right\} \sqrt{\frac{n}{x(1+x)}} \\
&\quad + \frac{5}{2} \frac{1}{x(1+x)} \left(\sum_{k=0}^{\infty} (k-nx)^2 v_{n,k}(x) \right)^{1/2} \left\{ \sum_{k=0}^{\infty} v_{n,k}(x)(n-1) \right. \\
&\quad \times \left. \int_0^{\infty} v_{n,k}(t)(t-x)^2 \, dt \right\}^{\alpha/2} + \left(\frac{2x}{n} \right)^{\alpha/2} \frac{1}{x(1+x)} \\
&\quad \times \left(\sum_{k=0}^{\infty} (k-nx)^2 v_{n,k}(x) \right)^{1/2} \\
&\quad \times \left\{ \sum_{k=0}^{\infty} v_{n,k}(x)(n-1) \int_0^{\infty} v_{n,k}(t)(t-x)^2 \, dt \right\}^{\alpha/4} \\
&\leqslant \left\{ \left(\frac{x(1+x)}{n} \right)^{\alpha/2} + 2n^{-\alpha} + (d(x, E))^{\alpha} \right\} \sqrt{\frac{n}{x(1+x)}} \\
&\quad + M \sqrt{\frac{n}{x(1+x)}} \left(\frac{x(1+x)}{n} + \frac{1}{n^2} \right)^{\alpha/2} \\
&\quad + M \left(\frac{2x}{n} \right)^{\alpha/2} \sqrt{\frac{n}{x(1+x)}} \left(\frac{x(1+x)}{n} + \frac{1}{n^2} \right)^{\alpha/4} \\
&\leqslant M' \sqrt{\frac{n}{x(1+x)}} \left\{ \left(\frac{x(1+x)}{n} \right)^{\alpha/2} + n^{-\alpha} + (d(x, E))^{\alpha} \right\}.
\end{aligned}
$$

Here we have used (2.1), Hölder's inequality several times, and the following formula:

$$
\sum_{k=0}^{\infty} (k-nx)^2 v_{n,k}(x) = nx(1+x).
$$

The proof of Lemma 2.3 is complete.

LEMMA 2.4. *Under the same conditions as in Lemma 2.3, we have*

$$|V_n'(f, x)| \leqslant M'n \left\{\left(\frac{x(1+x)}{n}\right)^{\alpha/2} + n^{-\alpha} + (d(x, E))^{\alpha}\right\}, \qquad x \in [0, \infty). \quad (2.10)$$

Proof. By (2.4), (2.7), (2.8), and (2.9), we have

$$\begin{aligned}
|V_n'(f, x)| &\leqslant n(n-1) \sum_{k=0}^{\infty} v_{n+1,k}(x) \int_0^{\infty} (v_{n,k+1}(t) + v_{n,k}(t)) \\
&\quad \times \left\{\left(\frac{x(1+x)}{n}\right)^{\alpha/2} + n^{-\alpha} + (d(x, E))^{\alpha} + 2\,|t-x|^{\alpha} \right.\\
&\quad \left. + \left(\frac{|t-x|}{n}\right)^{\alpha/2} + \left(\frac{2x(t-x)}{n}\right)^{\alpha/2}\right\} dt \\
&\leqslant 2n \left\{\left(\frac{x(1+x)}{n}\right)^{\alpha/2} + 2n^{-\alpha} + (d(x, E))\right\} \\
&\quad + 3n \sum_{k=0}^{\infty} v_{n+1,k}(x)(n-1) \\
&\quad \times \left\{\int_0^{\infty} (v_{n,k+1}(t) + v_{n,k}(t))(t-x)^2\, dt\right\}^{\alpha/2} \left\{\int_0^{\infty} (v_{n,k+1}(t) \right.\\
&\quad \left. + v_{n,k}(t)\, dt\right\}^{1-\alpha/2} + \left(\frac{2x}{n}\right)^{\alpha/2} n(n-1) \sum_{k=0}^{\infty} v_{n+1,k}(x) \\
&\quad \times \left\{\int_0^{\infty} (v_{n,k+1}(t) + v_{n,k}(t))(t-x)^2\, dt\right\}^{\alpha/4} \Bigg\} \\
&\quad \times \left\{\int_0^{\infty} (v_{n,k+1}(t) + v_{n,k}(t))\, dt\right\}^{1-\alpha/4} \\
&\leqslant 2n \left\{\left(\frac{x(1+x)}{n}\right)^{\alpha/2} + 2n^{-\alpha} + (d(x, E))^{\alpha}\right\} \\
&\quad + 3n \left\{\sum_{k=0}^{\infty} v_{n+1,k}(x)(n-1) \int_0^{\infty} (v_{n,k+1}(t) + v_{n,k}(t))(t-x)^2\, dt\right\}^{\alpha/2} \\
&\quad + \left(\frac{2x}{n}\right)^{\alpha/2} \times n \left\{\sum_{k=0}^{\infty} v_{n+1,k}(x)(n-1) \int_0^{\infty} (v_{n,k+1}(t) \right.\\
&\quad \left. + v_{n,k}(t))(t-x)^2\, dt\right\}^{\alpha/4}. \qquad (2.11)
\end{aligned}$$

By simple computation we have

$$(n-1)\sum_{k=0}^{\infty} v_{n+1,k}(x)\int_0^{\infty}(v_{n,k+1}(t)+v_{n,k}(t))(t-x)^2\,dt$$
$$\leqslant M\left(\frac{x(1+x)}{n}+\frac{1}{n^2}\right), \tag{2.12}$$

where M is independent of n and x.

Combining (2.11) and (2.12), we have

$$|V_n'(f,x)|\leqslant Mn\left\{\left(\frac{x(1+x)}{n}\right)^{\alpha/2}+n^{-\alpha}+(d(x,E))^{\alpha}\right\}, \qquad x\in[0,\infty).$$

The proof of Lemma 2.4 is complete.

Finally, we shall use the following inequality.

LEMMA 2.5. *Let* $0\leqslant\beta\leqslant 1$, $0\leqslant x_1<x_2<\infty$, $x_2-x_1\leqslant\frac{1}{4}$, $\varphi^2(t)=t(1+t)$. *Then we have*

$$\int_{x_1}^{x_2}(\varphi^2(u))^{(\beta-1)/2}\,du\leqslant M\frac{x_2-x_1}{(\max\{\varphi^2(x_2),\varphi^2(x_1)\}^{(1-\beta)/2}}. \tag{2.13}$$

Proof. Since $0\leqslant x_1<x_2$, then for $u\in[x_1,x_2]$, we have

$$\int_{x_1}^{x_2}(u(1+u))^{(\beta-1)/2}\,du\leqslant\int_{x_1}^{x_2}\frac{u^{(\beta-1)/2}\,du}{(3/4(1+x_2))^{(1-\beta)/2}}$$
$$\leqslant\frac{x_2^{(\beta+1)/2}-x_1^{(1-\beta)/2}}{\left(\frac{\beta-1}{2}+1\right)\left(\frac{3}{4}(1+x_2)\right)^{(1-\beta)/2}}$$
$$\leqslant\frac{x_2-x_1}{\left(\frac{\beta-1}{2}+1\right)\left(\frac{3}{4}(1+x_2)\right)^{(1-\beta)/2}x_2^{(1-\beta)/2}}$$
$$\leqslant M\frac{x_2-x_1}{(x_2(1+x_2))^{(1-\beta)/2}}.$$

Hence (2.13) holds and the proof of Lemma 2.5 is then complete.

3. PROOF OF THEOREM

With all the above preparations we can now prove our theorem. Here the commutativity of the Baskakov–Durrmeyer operators is crucial:

$$V_n(V_m f) = V_m(V_n f). \tag{3.1}$$

See [3].

Proof of Theorem. Sufficiency: Suppose that (1.4) holds. Then from the continuity of f we know that (1.4) holds for any $x \in [0, \infty)$ and $y \in \bar{E}$, the closure of the set E.

We prove (1.5). Let $x \in [0, \infty)$ and $x_0 \in \bar{E}$ be such that

$$|x - x_0| = d(x, E).$$

By (2.1) we have

$$\begin{aligned} |V_n(f, x) - f(x)| &\leqslant V_n(|f(t) - f(x_0)|, x) + |f(x) - f(x_0)| \\ &\leqslant V_n(M_f\,|t - x_0|^a, x) + M_f\,|x - x_0|^\alpha \\ &\leqslant V_n(M_f\,|t - x|^\alpha, x) + V_n(M_f\,|x - x_0|^\alpha, x) + M_f\,|x - x_0|^\alpha \\ &\leqslant 2M_f\,|x - x_0|^\alpha + M_f(V_n((t - x)^2, x))^{\alpha/2} \\ &\leqslant 2M_f \left\{ \left(\frac{x(1+x)}{n} + \frac{1}{n^2} \right)^{\alpha/2} + (d(x, E))^\alpha \right\}. \end{aligned}$$

Hence (1.5) holds and the proof of sufficiency is complete. We note that this direct part holds valid also for $\alpha = 1$.

Necessity. Suppose that (1.5) holds. Let $x \in (0, \infty)$, $y \in E$. We show that (1.4) is valid for a constant M_f.

If $|x - y| \geqslant \frac{1}{4}$, then we can easily obtain

$$|f(x) - f(y)| \leqslant 2\,\|f\|_\infty \leqslant 8\,\|f\|_\infty\,|x - y|^\alpha.$$

If $0 < |x - y| < \frac{1}{4}$, we choose $5 \leqslant n \in N$ such that

$$\begin{aligned} \frac{|x - y|}{2} &< \delta(n, x, y) := \max\left\{ \frac{1}{2^{n-2}}, \sqrt{\frac{x(1+x)}{2^{n-2}}}, \sqrt{\frac{y(1+y)}{2^{n-2}}} \right\} \\ &\leqslant |x - y|. \end{aligned} \tag{3.2}$$

This choice can always be realized because the sequence $\{\delta(n, x, y)\}_{n \in N}$ decreases monotonely to zero as n tends to infinity and it satisfies.

$$\delta(n, x, y) < \delta(n-1, x, y) \leqslant 2\delta(n, x, y), \qquad n \in N.$$

Under this choice, using (2.7) and (2.9), we have

$$\begin{aligned}
|f(x)-f(y)| \leqslant\, & |f(x)-V_{2^n}(f,x)| + |V_{2^n}(f,y)-f(y)| \\
& + |V_{2^n}(f-V_{2^{n-1}}(f),x) - V_{2^n}(f-V_{2^{n-1}}(f),y)| \\
& + |V_{2^n}(V_{2^{n-1}}f,x) - V_{2^n}(V_{2^{n-1}}f,y)| \\
\leqslant\, & M''_f \left(\frac{x(1+x)}{2^n}\right)^{\alpha/2} + (2^{-2n})^{\alpha/2} + (d(x,E))^{\alpha} \\
& + M''_f \left(\frac{y(1+y)}{2^n}\right)^{\alpha/2} + (2^{-2n})^{\alpha/2} + (d(y,E))^{\alpha} \\
& + M''_f V_{2^n} \left(\frac{t(1+t)}{2^{n-1}}\right)^{\alpha/2} + (2^{1-n})^{\alpha} + (d(t,E))^{\alpha}, x) \\
& + M''_f V_{2^n} \left(\left(\frac{t(1+t)}{2^{n-1}}\right)^{\alpha/2} + (2^{1-n})^{\alpha} + (d(t,E))^{\alpha}, y\right) \\
& + |V_{2^n}(V_{2^{n-1}}f,x) - V_{2^n}(V_{2^{n-1}}f,y)| \\
\leqslant\, & M''_f (5\,|x-y|^{\alpha} + 2(d(x,E))^{\alpha}) + M''_f V_{2^n} \left(\left(\frac{x(1+x)}{2^{n-1}}\right)^{\alpha/2}\right. \\
& + \left(\frac{|t-x|}{2^{n-1}}\right)^{\alpha/2} + (2^{1-n})^{\alpha} + \left(\frac{2x\,|t-x|}{2^{n-1}}\right)^{\alpha/2} + \frac{|t-x|^{\alpha}}{(2^{n-1})^{\alpha/2}} \\
& \left. + (d(x,E))^{\alpha} + |t-x|^{\alpha}, x\right) + M''_f V_{2^n} \left(\left(\frac{y(1+y)}{2^{n-1}}\right)^{\alpha/2}\right. \\
& + \left(\frac{|t-y|}{2^{n-1}}\right)^{\alpha/2} + (2^{1-n})^{\alpha} + \left(\frac{2y\,|t-y|}{2^{n-1}}\right)^{\alpha/2} \\
& \left. + \frac{|t-y|^{\alpha}}{(2^{n-1})^{\alpha/2}} + (d(y,E))^{a} + |t-y|^{\alpha}, y\right) \\
& + |V_{2^n}(V_{2^{n-1}}f,x) - V_{2^n}(V_{2^{n-1}}f,y)|.
\end{aligned}$$

By (3.1) we have

$$\begin{aligned}
V_{2^n}(V_{2^{n-1}}f,y) &= \sum_{j=3}^{n} (V_{2^j}(V_{2^{j-1}}f,y) - V_{2^{j-1}}(V_{2^{j-2}}f,y)) + V_4(V_2 f,y) \\
&= \sum_{j=3}^{n} V_{2^{j-1}}(V_{2^j}f - V_{2^{j-2}}f,y) + V_4(V_2 f,y).
\end{aligned}$$

Thus, using Lemma 2.1 and Lemma 2.2, we obtain

$$
\begin{aligned}
|f(x)-f(y)| \leqslant M_f'''(&|x-y|^\alpha + (d(x,E))^\alpha + (V_{2^n}((t-x)^2, x)^{\alpha/2} \\
&+ (V_{2^n}((t-y)^2, y))^{\alpha/2} + \left(\frac{x}{2^{n-1}}\right)^{\alpha/2} (V_{2^n}((t-x)^2, x))^{\alpha/4} \\
&+ \left(\frac{y}{2^{n-1}}\right)^{\alpha/2} (V_{2^n}((t-y)^2, y))^{\alpha/4}) + \left|\int_x^y V_4'(V_2 f, t)\, dt\right| \\
&+ \sum_{j=3}^{n} \left|\int_x^y |V'_{2^{j-1}}(V_{2^j} f - V_{2^{j-2}} f, t)|\, dt)\right| \\
\leqslant M_f''' &\left(|x-y|^\alpha + \sum_{j=3}^{n} I_j\right). \qquad (3.3)
\end{aligned}
$$

Here we have used the fact that $d(x,E) \leqslant |x-y|$, Hölder's inequality, and $|\int_x^y V_4'(V_2 f, t)\, dt| \leqslant c\, \|f\|_\infty\, |x-y| \leqslant c\, \|f\|_\infty\, |x-y|^\alpha$. We have also denoted

$$
I_j = \left|\int_x^y |V'_{2^{j-1}}(V_{2^j} f - V_{2^{j-2}} f, t)|\, dt\right|. \qquad (3.4)
$$

In order to derive (1.4), we need only estimate (3.4). Observe that

$$
|V_{2^j}(f,t) - V_{2^{j-2}}(f,t)| \leqslant 3M_f' \left\{\left(\frac{t(1+t)}{2^{j-1}}\right)^{\alpha/2} + (2^{j-1})^{-\alpha} + (d(t,E))^\alpha\right\}. \qquad (3.5)
$$

We use the local Bernstein-type inequalities and estimate (3.4) in three cases.

The first case is that $\delta(n,x,y) = 1/2^{n-2}$. Hence $2^{n-2} < 2/|x-y|$ by (3.2). We note that for $t \in [x,y]$ or $[y,x]$,

$$
d(t,E) \leqslant |t-y| \leqslant |x-y|. \qquad (3.6)
$$

Then we use Lemma 2.4 for 2^{j-1} and obtain

$$
\begin{aligned}
I_j &\leqslant \left|\int_x^y 3M_f'' 2^{j-1} \left\{\left(\frac{t(1+t)}{2^{j-1}}\right)^{\alpha/2} + (2^{j-1})^{-\alpha} + (d(t,E))^\alpha\right\} dt\right| \\
&\leqslant 3M_f'' \left\{|x-y|^{\alpha+1}\, 2^{j-1} + (2^{j-1})^{1-\alpha}\, |x-y| \right. \\
&\quad \left. + (2^{j-1})^{1-\alpha/2} \left|\int_x^y (t(1+t))^{\alpha/2}\, dt\right|\right\} \\
&\leqslant 3M_f'' \,\{|x-y|^{\alpha+1}\, 2^{j-1} + (2^{j-1})^{1-\alpha}\, |x-y| \\
&\quad + (2^{j-1})^{1-\alpha/2} (\max\{y(1+y), x(1+x)\})^{\alpha/2}\, |y-x|\}.
\end{aligned}
$$

Noting that $\max\{y(1+y), x(1+x)\} \leqslant 2^{-n+2}$, then we have

$$\sum_{i=3}^{n} I_j \leqslant 3M_f'' \left\{ 2^n |x-y|^{\alpha+1} + \frac{1}{1-2^{\alpha-1}} (2^{n-1})^{1-\alpha} |x-y| \right.$$
$$\left. + \frac{1}{1-2^{\alpha/2-1}} (2^{n-1})^{1-\alpha/2} (\max\{y(1+y), x(1+x)\})^{\alpha/2} |y-x| \right\}$$
$$\leqslant M_f''' |x-y|^{\alpha}. \tag{3.7}$$

Thus, we have estimated $\sum_{j=3}^{n} I_j$ when $\delta(n, x, y) = 1/2^{n-2}$.

On the other hand, when $\delta(n, x, y) = \max\{\varphi(x)/\sqrt{2^{n-2}}, \varphi(y)/\sqrt{2^{n-2}}\}$, we have $\max\{\varphi(x), \varphi(y)\} \geqslant 2^{(2-n)/2}$, and hence

$$\frac{|x-y|}{2} < \max\left\{\frac{\varphi(x)}{\sqrt{2^{n-2}}}, \frac{\varphi(y)}{\sqrt{2^{n-2}}}\right\} \leqslant |x-y|. \tag{3.8}$$

The second case of our estimate is $0 < \alpha < \frac{1}{2}$ and $\delta(n, x, y) = \max\{\varphi(x)/\sqrt{2^{n-2}}, \varphi(y)/\sqrt{2^{n-2}}\}$. In this case, the estimates are easy. In fact, by Lemma 2.3, Lemma 2.5, and (3.6) we have for $3 \leqslant j \leqslant n$,

$$I_j = \left| \int_x^y |V'_{2^{j-1}}(V_{2^j} f - V_{2^{j-2}} f, t)| \, dt \right|$$
$$\leqslant \left| \int_x^y M_f'' \sqrt{\frac{2^{j-1}}{t(1+t)}} \left\{ \left(\frac{t(1+t)}{2^{j-1}} \right)^{\alpha/2} + (2^{1-j})^{\alpha} + (d(t, E))^{\alpha} \right\} dt \right|$$
$$\leqslant M_f'' \left\{ (2^{j-1})^{(1-\alpha)/2} \left| \int_x^y (t(1+t))^{(\alpha-1)/2} \, dt \right| + \sqrt{2^{j-1}} ((2^{1-j})^{\alpha} \right.$$
$$\left. + |x-y|^{\alpha}) \left| \int_x^y (t(1+t))^{-1/2} \, dt \right| \right.$$
$$\leqslant M_f''' \{ |x-y| (\max\{\varphi(x), \varphi(y)\})^{\alpha-1} (2^{j-1})^{(1-\alpha)/2}$$
$$+ |x-y| (\max\{\varphi(x), \varphi(y)\})^{-1} (2^{j-1})^{1/2-\alpha}$$
$$+ |x-y|^{1+\alpha} (\max\{\varphi(x), \varphi(y)\})^{-1} (2^{j-1})^{1/2} \}.$$

Thus, taking sums over j we get from (3.8)

$$\sum_{j=3}^{n} I_j \leqslant M_f''' \left\{ |x-y| (\max\{\varphi(x), \varphi(y)\})^{\alpha-1} (2^{n-1})^{(1-\alpha)/2} \frac{1}{1-2^{(\alpha-1)/2}} \right.$$
$$+ |x-y| (\max\{\varphi(x), \varphi(y)\})^{-1} (2^{n-1})^{1/2-\alpha} \frac{1}{1-2^{\alpha-1/2}}$$
$$\left. + |x-y|^{1+\alpha} (\max\{\varphi(x), \varphi(y)\})^{-1} (2^{n-1})^{1/2} \frac{1}{1-1/\sqrt{2}} \right\}$$

$$\leqslant M_f''' \left\{ |x-y| \left(\max \left\{ \frac{\varphi(x)}{2^{(n-2)/2}}, \frac{\varphi(y)}{2^{(n-2)/2}} \right\} \right)^{\alpha-1} + |x-y| \left(\max \left\{ \frac{\varphi(x)}{2^{(n-2)/2}}, \frac{\varphi(y)}{2^{(n-2)/2}} \right\} \right)^{-1} (2^{2-n})^{\alpha} + |x-y|^{1+\alpha} \left(\max \left\{ \frac{\varphi(x)}{2^{(n-2)/2}}, \frac{\varphi(y)}{2^{(n-2)/2}} \right\} \right)^{-1} \right\}$$

$$\leqslant 3M_f'' \, |x-y|^{\alpha}, \tag{3.9}$$

where M_f''' is constant depending only on α and f.

We have then completed the estimate of $\sum_{j=3}^{n} I_j$ in the second case.

The final case is $\frac{1}{2} \leqslant \alpha < 1$ and $\delta(n, x, y) = \max\{\varphi(x)/\sqrt{2^{n-2}}, \varphi(y)/\sqrt{2^{n-2}}\}$.

The proof of this case is somewhat difficult.

Let $3 \leqslant j \leqslant n$. If $\max\{\varphi(x), \varphi(y)\} \geqslant 2^{(2-j)/2}$, i.e., $2^{(j-2)/2} \max\{\varphi(x), \varphi(y)\} \geqslant 1$.

Then by Lemma 2.3, Lemma 2.5, we have

$$I_j \leqslant \left| \int_x^y M_f'' \sqrt{\frac{2^{j-1}}{t(1+t)}} \left\{ \left(\frac{t(1+t)}{2^{j-1}} \right)^{\alpha/2} + (2^{1-j})^{\alpha} + (d(t, E))^{\alpha} \right\} dt \right|$$

$$\leqslant M_f''(2^{j-1})^{(1-\alpha)/2} \, |x-y| \, (\max\{\varphi(x), \varphi(y)\})^{\alpha-1} + M_f''(2^{j-1})^{1/2-\alpha} \, |x-y| (\max\{\varphi(x), \varphi(y)\})^{-1} + M_f'' \, |x-y|^{\alpha} \, 2^{(j-1)/2} \, |x-y| (\max\{\varphi(x), \varphi(y)\})^{-1}$$

$$\leqslant M_f' \, |x-y|^{1+\alpha} \frac{2^{(j-2)/2}}{\max\{\varphi(x), \varphi(y)\}} + M_f' \, |x-y| \frac{(2^{j-2})^{(1-\alpha)/2}}{\max\{\varphi(x), \varphi(y)\})^{1-\alpha}}. \tag{3.10}$$

On the other hand, if $\max\{\varphi(x), \varphi(y)\} < 2^{(2-j)/2}$, i.e., $2^{j-2} < (2^{(j-2)/2}/\max\{\varphi(x), \varphi(y)\})$.

Then by Lemma 2.4, we have

$$I_j \leqslant \left| \int_x^y M_f'' 2^{j-1} \left\{ \left(\frac{t(1+t)}{2^{j-1}} \right)^{\alpha/2} + (2^{1-j})^{\alpha} + (d(t, E))^{\alpha} \right\} dt \right|$$

$$\leqslant M_f''(2^{j-1})^{1-\alpha/2} \, (\max\{\varphi(x), \varphi(y)\})^{\alpha} \, |x-y| + M_f'' \, |x-y| (2^{j-1})^{1-\alpha} + M_f'' \, |x-y|^{1+\alpha} \, (2^{j-1})$$

$$\leqslant M_f''' \, |x-y| (2^{j-2})^{(1-\alpha)/2} \, (\max\{\varphi(x), \varphi(y)\})^{\alpha-1} + M_f''' \, |x-y|^{1+\alpha} \, (2^{j-2})^{1/2} \, (\max\{\varphi(x), \varphi(y)\})^{-1}. \tag{3.11}$$

Combining (3.10) and (3.11), we have for $3 \leqslant j \leqslant n$

$$I_j \leqslant M_f''' \, |x-y|^{1+\alpha} \, (2^{j-2})^{1/2} \, (\max\{\varphi(x), \varphi(y)\})^{-1} + M_f''' \, |x-y| (2^{j-2})^{(1-\alpha)/2} \, (\max\{\varphi(x), \varphi(y)\})^{\alpha-1}.$$

By taking sums over j, we have in the final case

$$\sum_{j=3}^{n} I_j \leqslant M_f''' \, |x-y|^{1+\alpha} \left(\max \left\{ \frac{\varphi(x)}{2^{(n-2)/2}}, \frac{\varphi(y)}{2^{(n-2)/2}} \right\} \right)^{-1} + M_f''' \, |x-y| \left(\max \left\{ \frac{\varphi(x)}{2^{(n-2)/2}}, \frac{\varphi(y)}{2^{(n-2)/2}} \right\} \right)^{\alpha-1} \leqslant 2M_f''' \, |x-y|^{\alpha}. \tag{3.12}$$

Thus, combining (3.7), (3.9), and (3.12) we have for all the cases

$$\sum_{j=3}^{n} I_j \leqslant C_\alpha M_f'' \, |x-y|^{\alpha},$$

where C_α is a constant depending only on α. M_f'' is also a constant depending only on f and α.

Using this estimate in (3.3), we obtain for any $x \in [0, \infty)$, $y \in E$,

$$|f(x) - f(y)| \leqslant M_f \, |x-y|^{\alpha}.$$

Thus, we complete the proof of the theorem.

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